Fixed point theorems on partial randomness

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Abstract

In this talk, we show the following fixed point theorem on partial randomness, from the point of view of algorithmic randomness.

Theorem [fixed point theorem on partial randomness]
For every $T \in (0,1)$, if $\Omega(T)$ is a computable real number, then
(i) T is right-computable and not left-computable,
(ii) T is weakly Chaitin T-random and T-compressible,
(iii) $\lim_{n\to\infty} \frac{H(T\restriction n)}{n} = T$. $\Longrightarrow$ The compression rate of T equals to T. □

After that, we introduce variants of this theorem, and investigate their properties and relation.

Preliminaries: Program-size Complexity

- $\{0,1\}^* := \{\lambda, 0, 1, 00, 01, 10, 11, 000, \dots\}$. The set of finite binary strings.
- For any $s \in \{0,1\}^*$, $|s|$ denotes the *length* of s.
- Let $V \subset \{0,1\}^*$. We say V is *prefix-free* if for any distinct s and $t \in V$, s is not a prefix of t.

For example $\{0, 10\}$: prefix-free $\{0, 01\}$: not prefix-free

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Let U be a universal self-delimiting Turing machine.

Dom U, i.e., the domain of definition of U, is a prefix-free set.

Definition The *program-size complexity* (or *Kolmogorov complexity*) $H(s)$ of $s \in \{0,1\}^*$ is defined by

$$H(s) := \min\left\{ |p| \;\middle|\; p \in \{0,1\}^* \;\&\; U(p) = s \right\}.$$

$H(s)$: The length of the shortest input for the universal self-delimiting Turing machine U to output s. $\Longrightarrow$ $H(s)$: The degree of randomness of s.

Preliminaries: Randomness of Real Number

Definition For any $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}^+$, we denote by $\alpha \restriction n$ the first n bits of the base-two expansion of $\alpha - \lfloor \alpha \rfloor$, i.e., the fractional part of α.

Definition [weak Chaitin randomness, Chaitin 1975]
We say $\alpha \in \mathbb{R}$ is *weakly Chaitin random* if $n \le H(\alpha \restriction n) + O(1)$,
i.e., any prefix of the base-two expansion of α cannot be compressed by H.

This notion is equivalent to Martin-Löf randomness (Schnorr).

Definition [Chaitin's halting probability Ω, Chaitin 1975]

$$\Omega := \sum_{p \in \operatorname{Dom} U} 2^{-|p|}.$$

Theorem [Chaitin 1975] Ω is weakly Chaitin random.

Preliminaries: Partial Randomness of Real Number

The partial randomness (degree of randomness) of a real number can be characterized by a real number.

Definition [weak Chaitin T-randomness, Tadaki 2002] Let $T \in [0,1]$.
We say $\alpha \in \mathbb{R}$ is *weakly Chaitin T-random* if $Tn \leq H(\alpha \restriction n) + O(1)$.

In the case of $T = 1$, the weak Chaitin T-randomness results in the weak Chaitin randomness.

Definition [T-compressibility] Let $T \in [0,1]$.
We say $\alpha \in \mathbb{R}$ is *T-compressible* if $H(\alpha \restriction n) \leq Tn + o(n)$,

which is equivalent to $\limsup_{n\to\infty} \frac{H(\alpha \restriction n)}{n} \leq T$.

Remark If $\alpha \in \mathbb{R}$ is weakly Chaitin T-random and T-compressible, then

$$\lim_{n\to\infty} \frac{H(\alpha \restriction n)}{n} = T.$$

The *compression rate* of α by program-size complexity equals to T.

⟨The converse does not necessarily hold.⟩

Preliminaries: Generalization of Ω

Definition [generalization of Chaitin's Ω, Tadaki 1999]

$$\Omega(T) := \sum_{p \in \mathrm{Dom}\, U} 2^{-\frac{|p|}{T}} \qquad (T > 0).$$

$\Omega(1) = \Omega$.

Theorem [Tadaki 1999] Let $T \in \mathbb{R}$.

(i) If $0 < T \leq 1$ and T is computable, then $\Omega(T)$ is weakly Chaitin T-random and T-compressible. $\Longrightarrow$ The compression rate of $\Omega(T)$ equals to T.

(ii) If $1 < T$, then $\Omega(T)$ diverges to ∞.

Here, T is called *computable* if the mapping $\mathbb{N}^+ \ni n \mapsto T \restriction n$ is a total recursive function.

Fixed Point Theorem on Partial Randomness

Theorem [fixed point theorem on partial randomness, Tadaki, CiE 2008]

For every $T \in (0,1)$, if $\Omega(T)$ is a computable real number, then

(i) T is right-computable and not left-computable,

(ii) T is weakly Chaitin T-random and T-compressible,

(iii) $\lim_{n\to\infty} \dfrac{H(T \restriction n)}{n} = T$. $\Longrightarrow$ The compression rate of T equals to T itself.

□

Here, a real α is called *right-computable* if the set $\{\, r \in \mathbb{Q} \mid \alpha < r \,\}$ is r.e., and α is called *left-computable* if $-\alpha$ is right-computable.

Proof of Fixed Point Theorem

Theorem [fixed point theorem on partial randomness,] [posted again]

For every $T \in (0,1)$, if $\Omega(T)$ is a computable real number, then

(i) T is right-computable and not left-computable,

(ii) T is weakly Chaitin T-random and T-compressible,

(iii) $\lim_{n\to\infty} \frac{H(T \restriction n)}{n} = T$. □

Lemma [upper bound I] For every $T \in (0,1)$, if $\Omega(T)$ is right-computable then T is also right-computable. □

Lemma [upper bound II] For every $T \in (0,1)$, if $\Omega(T)$ is left-computable and T is right-computable, then T is T-compressible. □

Lemma [lower bound] For every $T \in (0,1)$, if $\Omega(T)$ is right-computable then T is weakly Chaitin T-random. □

Proofs of the three lemmas

Lemma [upper bound I] For every $T \in (0,1)$, if $\Omega(T)$ is right-computable then T is also right-computable.

Proof) For each $k \in \mathbb{N}^+$ and $x \in (0,1)$, let $\omega_k(x) = \sum_{i=1}^{k} 2^{-|p_i|/x}$, where $p_1, p_2, p_3, \ldots\ldots$ is a particular recursive enumeration of $\operatorname{Dom} U$.

Then we see that, for every $r \in \mathbb{Q} \cap (0,1)$, $T < r$ if and only if there exists $k \in \mathbb{N}^+$ such that $\Omega(T) < \omega_k(r)$. This is because $\Omega(x)$ is an increasing function of $x \in (0,1]$ and $\lim_{k\to\infty} \omega_k(r) = \Omega(r)$.

Since $\Omega(T)$ is right-computable,

the set $\{\, r \in \mathbb{Q} \cap (0,1) \mid \exists k \in \mathbb{N}^+ \ \Omega(T) < \omega_k(r) \,\}$ is r.e. and therefore

the set $\{\, r \in \mathbb{Q} \cap (0,1) \mid T < r \,\}$ is also r.e. □

Lemma [upper bound II] For every $T \in (0,1)$, if $\Omega(T)$ is left-computable and T is right-computable, then T is T-compressible.

Proof) Omitted. □

Lemma [lower bound] For every $T \in (0,1)$, if $\Omega(T)$ is right-computable then T is weakly Chaitin T-random.

Proof) The following procedure calculates a partial recursive function $\Psi\colon \{0,1\}^* \to \{0,1\}^*$ such that $Tn - Tc < H(\Psi(T \restriction n))$. The lemma follows from $H(\Psi(T \restriction n)) \leq H(T \restriction n) + O(1)$. Let $\omega_k(x) = \sum_{i=1}^{k} 2^{-|p_i|/x}$.

Procedure: Given $T \restriction n$, one can effectively find k_0 which satisfies

$$\Omega(T) < \omega_{k_0}(0.(T \restriction n) + 2^{-n}).$$

This is possible because $\Omega(x)$ is an increasing function of x, $\lim_{k\to\infty} \omega_k(r) = \Omega(r)$ for every $r \in \mathbb{Q} \cap (0,1)$, and $\Omega(T)$ is right-computable. It follows that

$$\sum_{i=k_0+1}^{\infty} 2^{-\frac{|p_i|}{T}} = \Omega(T) - \omega_{k_0}(T) < \omega_{k_0}(0.(T \restriction n) + 2^{-n}) - \omega_{k_0}(T) < 2^{c-n}.$$

Hence, for every $i > k_0$, $2^{-\frac{|p_i|}{T}} < 2^{c-n}$ and therefore $Tn - Tc < |p_i|$. Thus, by calculating the set $\{\, U(p_i) \mid i \leq k_0 \,\}$ and picking any one finite binary string s which is not in this set, one can then obtain $s \in \{0,1\}^*$ such that $Tn - Tc < H(s)$. $\square$

Remark on the sufficient condition in the fixed Point Theorem

Theorem [fixed point theorem on partial randomness] [posted again]

For every $T \in (0,1)$, if $\Omega(T)$ is computable then T is weakly Chaitin T-random and T-compressible. $\square$

Note that $\Omega(x)$ is a strictly increasing continuous function of $x \in (0,1)$, and the set of all computable real numbers is dense in $\mathbb{R}$. Thus,

Theorem The set $\{T \in (0,1) \mid \Omega(T) \text{ is computable}\}$ is dense in $(0,1)$. $\square$

Corollary [density of the fixed points]

The set $\{T \in (0,1) \mid T$ is weakly Chaitin T-random and T-compressible$\}$ is dense in $(0,1)$. $\square$

At this point, the following question would arise naturally:

Question **Is this sufficient condition, i.e., the computability of $\Omega(T)$, also necessary for T to be a fixed point ?**

Answer **Completely not !! (as we can see through the following argument)**

Thermodynamic Quantities in AIT: Definition

The thermodynamic quantities in AIT (algorithmic information theory) is introduced in the following manner.

Definition Let $q_1, q_2, q_3, \ldots\ldots$ be an arbitrary enumeration of $\mathrm{Dom}\, U$. □

Note that the results of this talk are independent of the choice of $\{q_i\}$.

Definition [thermodynamic quantities in AIT, Tadaki, CiE 2008] Let $T > 0$.

(i) Partition Function: $Z(T) := \lim_{k\to\infty} Z_k(T)$, where $Z_k(T) = \sum_{i=1}^{k} 2^{-\frac{|q_i|}{T}}$.

(ii) Free Energy: $F(T) := \lim_{k\to\infty} F_k(T)$, where $F_k(T) = -T \log_2 Z_k(T)$.

(ii) Energy: $E(T) := \lim_{k\to\infty} E_k(T)$, where $E_k(T) = \dfrac{1}{Z_k(T)} \sum_{i=1}^{k} |q_i|\, 2^{-\frac{|q_i|}{T}}$.

(iii) Entropy: $S(T) := \lim_{k\to\infty} S_k(T)$, where $S_k(T) = \dfrac{E_k(T) - F_k(T)}{T}$. □

Remark (i) $Z(T) = \Omega(T)$. (ii) The real T corresponds to "temperature".

Thermodynamic Quantities in AIT: Properties

The thermodynamic quantities $F(T)$, $E(T)$, and $S(T)$ has the almost same randomness properties as $\Omega(T)$, i.e., $Z(T)$.

Theorem [free energy $F(T)$] Let $T \in \mathbb{R}$.

(i) If $0 < T \leq 1$ and T is computable, then $F(T)$ converges to a real number which is weakly Chaitin T-random and T-compressible. (same as for $\Omega(T)$)
(ii) If $1 < T$, then $F(T)$ diverges to $-\infty$. □

Definition We say $\alpha \in \mathbb{R}$ is *Chaitin T-random* if $\lim_{n\to\infty} H(\alpha\restriction n) - Tn = \infty$.

Theorem [energy $E(T)$] Let $T \in \mathbb{R}$.

(i) If $0 < T < 1$ and T is computable, then $E(T)$ converges to a real number which is Chaitin T-random and T-compressible.
(ii) If $1 \leq T$, then $E(T)$ diverges to ∞. □

Theorem [entropy $S(T)$] Let $T \in \mathbb{R}$.

(i) If $0 < T < 1$ and T is computable, then $S(T)$ converges to a real number which is Chaitin T-random and T-compressible.
(ii) If $1 \leq T$, then $S(T)$ diverges to ∞. □

Thermodynamic Quantities in AIT: Fixed Point Theorems

In the fixed point theorem, $\Omega(T)$ can be replaced by each of the thermodynamic quantities $F(T)$, $E(T)$, and $S(T)$.

Theorem [fixed point theorem by the free energy $F(T)$]

For every $T \in (0, 1)$, if $F(T)$ is computable, then

(i) T is right-computable and not left-computable,

(ii) T is weakly Chaitin T-random and T-compressible. □

This theorem has the exactly same form as for $\Omega(T)$.

Theorem [fixed point theorem by the energy $E(T)$]

For every $T \in (0, 1)$, if $E(T)$ is computable, then

(i) T is right-computable and not left-computable,

(ii) T is Chaitin T-random and T-compressible. □

Theorem [fixed point theorem by the entropy $S(T)$]

For every $T \in (0, 1)$, if $S(T)$ is computable, then

(i) T is right-computable and not left-computable,

(ii) T is Chaitin T-random and T-compressible. □

Proof of the fixed point theorem by free energy $F(T)$

Theorem [general form of fixed point theorem] Let $f\colon (0,1) \to \mathbb{R}$. Suppose that f is a strictly increasing function and there is $g\colon (0,1) \times \mathbb{N}^+ \to \mathbb{R}$ which satisfies the following conditions:

(i) $\forall\, T \in (0,1)\ \ \lim_{k\to\infty} g(T,k) = f(T)$.

(ii) The mapping $\mathbb{Q} \times (\mathbb{Q} \cap (0,1)) \ni (r,k) \mapsto g(r,k)$ is computable.

(iii) $\forall\, T \in (0,1)\ \ \exists\, k_0 \in \mathbb{N}^+\ \ \exists\, a,b \in \mathbb{N}\ \ \forall\, k \geq k_0$

$$2^{-\frac{|p_{k+1}|}{T}-a} \leq g(T,k+1) - g(T,k) \leq 2^{-\frac{|p_{k+1}|}{T}+b}.$$

(iv) $\forall\, T \in (0,1)\ \ \exists\, t \in (T,1)\ \ \exists\, k_0 \in \mathbb{N}^+\ \ \exists\, c,d \in \mathbb{N}\ \ \forall\, k \geq k_0\ \ \forall\, x \in (T,t)$

$$2^{-c}(x-T) \leq g(x,k) - g(T,k) \leq 2^{d}(x-T).$$

(v) $\forall\, t_1, t_2 \in (0,1)$ with $t_1 < t_2\ \exists\, k_0 \in \mathbb{N}^+\ \forall\, k \geq k_0\ \forall\, x \in [t_1,t_2]\ g(x,k) \leq f(x)$.

(vi) $\forall\, k \in \mathbb{N}^+\ \ \forall\, T \in (0,1)\ \ \lim_{x\to T+0} g(x,k) = g(T,k)$.

Then, for every $T \in (0,1)$, if $f(T)$ is computable, then T is weakly Chaitin T-random and T-compressible. □

Proof of the fixed point theorem by free energy $F(T)$

Theorem [fixed point theorem by free energy $F(T)$] [posted again]

For every $T \in (0,1)$, if $F(T)$ is a computable real number, then T is weakly Chaitin T-random and T-compressible. □

A portion of the proof:

Using the mean value theorem and the lemma below,

$$S_k(T)(x-T) \leq F_k(T) - F_k(x) \leq S_k(t)(x-T)$$

for every $k \in \mathbb{N}^+$ and every $T, x, t \in (0,1)$ with $T < x < t$. On the other hand, for every $T \in (0,1)$, there exists $k_0 \in \mathbb{N}^+$ such that, for every $k \geq k_0$,

$$0 < S_{k_0}(T) \leq S_k(T) \leq S(T).$$

Lemma [thermodynamic relation] Let $T \in (0,1)$ and $k \in \mathbb{N}^+$.

(i) $F_k'(T) = -S_k(T)$, $E_k'(T) = C_k(T)$, and $S_k'(T) = C_k(T)/T$.

(ii) $F'(T) = -S(T)$, $E'(T) = C(T)$, and $S'(T) = C(T)/T$.

(iii) $S_k(T), C_k(T) \geq 0$ and $S(T), C(T) > 0$. □

Relation between the sufficient conditions of FPTs I

Theorem There does not exist $T \in (0,1)$ such that both $\Omega(T)$ and $F(T)$ are computable.

Proof)
Contrarily, assume that both $\Omega(T)$ and $F(T)$ are computable for some $T \in (0,1)$. Since the statistical mechanical relation $F(T) = -T \log_2 \Omega(T)$ holds,

$$T = -\frac{F(T)}{\log_2 \Omega(T)}.$$

Thus, T is computable, and therefore $\Omega(T)$ is weakly Chaitin T-random, i.e., $Tn \leq H((\Omega(T)) \restriction n) + O(1)$. However, this is impossible, since $\Omega(T)$ is computable and therefore $H((\Omega(T)) \restriction n) \leq 2 \log_2 n + O(1)$. Thus we have a contradiction. □

$$\underbrace{\{T \in (0,1) \mid \Omega(T) \text{ is computable}\}}_{\text{dense in } (0,1)} \cap \underbrace{\{T \in (0,1) \mid F(T) \text{ is computable}\}}_{\text{dense in } (0,1)} = \emptyset.$$

In particular, this shows that the computability of $\Omega(T)$ is not a necessary condition for T to be a fixed point in the fixed point theorem by $\Omega(T)$.

Relation between the sufficient conditions of FPTs II

Theorem There does not exist $T \in (0,1)$ such that all of $\Omega(T)$, $E(T)$, and $S(T)$ are computable.

Proof) Use the statistical mechanical relation

$$S(T) = \frac{E(T)}{T} + \log_2 \Omega(T).$$

□

Theorem There does not exist $T \in (0,1)$ such that all of $F(T)$, $E(T)$, and $S(T)$ are computable.

Proof) Use the thermodynamic relation

$$S(T) = \frac{E(T) - F(T)}{T}.$$

□

Some other property of the sufficient condition in FPTs

Using the fixed point theorem by $\Omega(T)$, some property of the computability of $\Omega(T)$ is derived.

Let $T \in (0,1)$ and $a \in (0,1]$. Assume that a is computable.

$$\Omega(aT) \text{ is computable } \Rightarrow \lim_{n\to\infty} \frac{H((aT)\restriction n)}{n} = aT \Rightarrow \lim_{n\to\infty} \frac{H(T\restriction n)}{n} = aT.$$

by FPT by $H((aT)\restriction n) = H(T\restriction n) + O(1)$

Theorem $S_a \cap S_b = \emptyset$ for any distinct computable real numbers $a, b \in (0,1]$, where $S_a = \{T \in (0,1) \mid \Omega(aT) \text{ is computable}\}$. □

Example For every $T \in (0,1)$, if $\Omega(T)$ is computable, then for each integer $n \geq 2$, $\Omega(T/n)$ is not computable. Namely,

for every $T \in (0,1)$, if the sum $\sum_{p\in \mathrm{Dom}\, U} 2^{-|p|/T}$ is computable, then its power sum $\sum_{p\in \mathrm{Dom}\, U} \left(2^{-|p|/T}\right)^n$ is not computable for every integer power $n \geq 2$. □

Summary

In this talk, we introduced and showed the following fixed point theorem on partial randomness, from the point of view of algorithmic randomness.

Theorem [fixed point theorem on partial randomness]
For every $T \in (0,1)$, if $\Omega(T)$ is a computable real number, then
(i) T is right-computable and not left-computable,
(ii) T is weakly Chaitin T-random and T-compressible,
(iii) $\lim_{n\to\infty} \frac{H(T \restriction n)}{n} = T$. □

After that, we introduced several variants of this theorem, and investigate their properties and relation. In particular, we showed that the sufficient condition for T to be a fixed point is not a necessary condition in the fixed point theorems.